There are More Than Part-Whole Strategies at Work in Understanding Non-Equal-Parts Fraction-Area-Models

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In this study, 88 Grade 6 students' responses to three part-whole tasks are analysed in terms of Kieren's (1988, 1993) four-part model for fractions- measure, quotient, operator, and ratio. Using data from interviews, their strategies were also analysed using traditional part-whole explanations, including identifying the count-and-match misconception which was found to have distinct variations.

Statement of the Problem

If a part-whole sub-construct is used as a conceptual model to analyse students' performance on fractions tasks, then simple categories are established: correct part-whole understanding, the count-and-match misconception, and other incorrect part-whole strategies. The intent of this paper is to examine whether analysing students' performance on part-whole tasks in terms of Kieren's (1993) four sub-constructs - measure, quotient, operator, ratio- enables more precise interpretations of students' strategies.

Context of the Problem in the Research Literature

The Part-Whole Count-and-Match Misconception and Kieren's Four-Part Model

Traditional fraction instruction in the primary school often emphasises a part-whole understanding of fractions; a whole is pre-divided into equal parts and students have to identify the fractional part shaded (Carrahar, 1996; Gould, 2005).

Over the last four decades researchers have developed explanations of this part-whole concept and related curriculum models. Three independent branches have developed: the Rational Number Project (Behr, Lesh, Post & Silver, 1983; Behr, Khoury, Harel, Post, & Lesh 1997) based on Kieren (1980) who initially identified part-whole as one of five subconstructs- part-whole, measure, operator, quotient, and ratio; the Dutch realistic mathematics education group who developed a curriculum that explored part-whole understandings in the context of fair sharing, (Streefland, 1991); and Steffe (2002) who developed an explanation of part-whole understandings as partitive unit fractional schemes.

A count-and-match misconception has been observed when students, drilled in a procedure, count the shaded pieces (numerator) then count all the pieces (denominator), without regard for the size of the parts (Kieren, 1993; Carrahar, 1996; Gould, 2005).

The three main explanatory schemes and their associated curriculum models address this count-and-match misconception in different ways. The Rational Number Project curriculum advocates careful assessment to check whether the student's part-whole knowledge is robust (Cramer, Behr, Post and Lesh, 1997), followed by intervention if necessary. Streefland (1991) does not explicitly describe the misconception, but the first cluster of activities in this Dutch curriculum, serving up and distributing, exposes students to unequal parts, and aims to develop correct fractional language. For these students, working with non-equal parts is routine and counting-and-matching would not be described as a specific misconception needing to be uncovered, but rather as incomplete part-whole understanding that every student moves through. For researchers building on

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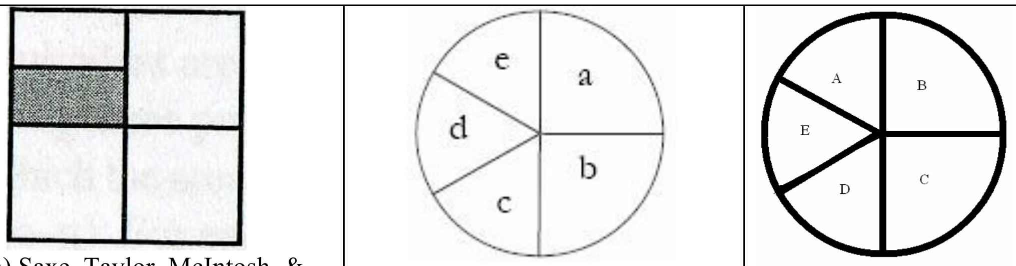
Steffe's work, counting and matching in non-equal-area examples is a natural conjecture that would be self-corrected by the action of iteration because misnamed non-equal pieces will not iterate into the whole successfully. A quarter misnamed a third in a stick divided into three pieces- a half and two quarters – will iterate four times rather than three times, generating perturbation (Norton, 2008). The second and third approaches have developed curriculum models in which counting-and-matching is a self-correcting phase, not a misconception.

Kieren's response to the count-and-match misconception was didactic and dramatic. In prior revisions of his model of fraction knowledge, he had progressively eradicated rote procedures; algorithms were removed in 1980. This revision continued, and part-whole, as a sub-construct, was removed in 1988. In Kieren's (1993) further explanation of his four part model - measure, quotient, operator, ratio; underpinned by the actions of partitioning, equivalence, and unit forming- part-whole understandings can be understood as part of the other sub-constructs, and within their explanatory boundaries. For Kieren, the measure and quotient sub-constructs provide conceptually richer ways of explaining the non-procedural part-whole examples used by the researchers in the Rational Number Project (p.57). Like the Dutch and Steffe, Kieren (1993) demanded better classroom activities and explanations in order to make the count-and-match misconception a self-limiting stage, but unlike them, he presented the theoretical shift of removing part-whole as a sub-construct in order to enable such a curriculum.

However, Kieren's four-part model has not been taken up in the research literature. Much of the research that uses Kieren's sub-constructs that followed his 1988 and 1993 articles still used the five-part model which gives part-whole its own descriptive category, rather than Kieren's later model which requires "part-whole" problems to be described in terms of the four sub-constructs: measure, quotient, operator, and ratio. One reason for this was the dominance of the Rational Number Project which used the 5-part model for historical reasons – their data collection and theoretical framework had been based on Kieren's earlier work. Even researchers who quote aspects of Kieren's 1993 article often still use a five part model (e.g. Lamon, 2007). This would indicate that there is still a strong desire in the research community to conceptualise part-whole as an equal concept or context to the four other sub-constructs; measure, quotient, operator and ratio.

Students' Performance on Non-Equal-Parts Fraction-Area-Problems in the Literature

Non-equal-parts fraction-area-problems are a context where the count-and-match misconception becomes apparent because the denominator should relate to the size of the piece not the number of unequal pieces (Saxe, Taylor, McIntosh, & Gearhart, 2005). Using a repeated halving model, only 9% of 384 Grade 4, 5, and 6 students identified a shaded piece as 1/8, while 25% on this pen and paper test identified it as 1/5 (Saxe et al., 2005), see figure 1a. Of 20 Grade 6 students, only two were correct at identifying part c as 1/6, while five wrote 1/5 (Stewart, 2005), see figure 1b. Using a pie task adapted from the same source (Cramer et al., 1997), Clarke, Roche, and Mitchell (2007) reported on 323 interviews with Grade 6 students in which 42.7% gave the correct answer of 1/6 for the smaller piece, A, with 13.6% answering 1/5, see figure 1c. There is evidence that correctly identifying parts in non-equal area models is difficult, and the count-and-match



(a) Saxe, Taylor, McIntosh, & Gearhart, 2005 | (b) Stewart, 2005 | (c) Clarke, Roche, Mitchell, 2007

Figure 1. Non-equal-parts tasks in the research literature

misconception is noted and appears evident in the responses of up to a quarter of the students in these studies.

Critique of Methodologies in the Research Literature

In many research projects, tasks are assigned as being a *part-whole* task, an *operator* task, or a *measure* task (e.g. Moseley, 2005). This is particularly true if data collection with pen and paper tests is involved. It is perhaps more correct to think of tasks as having a part-whole context, or operator, or measure context. However, students' performance can only be analysed in terms of the strategy they offer, which may or may not be the same as the nominal context of the task.

There is a danger of incorrectly assigning strategies to students' responses. In all three examples above, the answer of 1/5 is interpreted as evidence of a count-and-match misconception. Verification of this categorisation would require additional evidence.

Much research has looked at students' performance on tasks from one sub-construct. Some researchers have looked at students' performance across tasks from different sub-constructs (e.g. Moseley, 2005; Charalambous & Pitta-Pantazi, 2007). However, there is a lack of research analysing individual tasks in terms of the range of responses that they generate which can be categorised under different sub-constructs.

Methodology

In order to gain more insight into students' strategies, a one-to-one task-based interview was developed in which students were offered fraction tasks and their response probed with, *and how did you work that out*, to elicit more of their reasoning. They were not told whether their answers were correct or incorrect, and this interview protocol was intended to demonstrate interest in, and focus on, the students' strategies. A range of responses was desired, so a sample of 88 Grade 6 students from three schools, of different socio-economic status, in metropolitan Melbourne were interviewed between February and June, 2008. Each interview was audio-recorded and just over three quarters of the sample were also video-recorded. Notes were taken during each interview and recorded on a record sheet. The students' responses to some tasks were transcribed.

Two tasks are reported on here: the Fraction Pie task, see Figure 2a and the Fraction Sort task, see Figure 2b, 2c, 2d. In the Fraction Pie task, adapted from Cramer et al. (1997), the students were asked, *what fraction of the circle is part A* (and then Part B) with the interviewer indicating the parts with her finger. The Fraction Sort task required the student to look at a representation of a fraction and then place it in one of four piles - a quarter, a sixth, two-thirds, or other. At the beginning of the task the student was asked to offer a

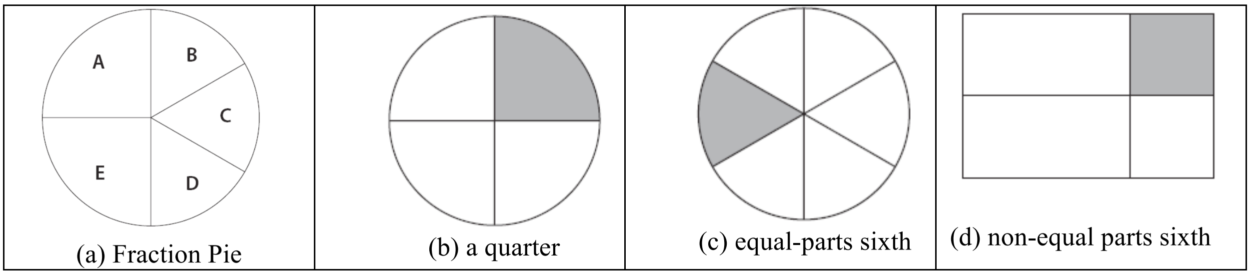


Figure 2. Fraction Pie and Fraction Sort task cards

brief explanation for each card as they placed it in the chosen pile. Of the 24 cards offered, three are discussed here. In both tasks, the diagrams were presented on laminated cards. No data was collected on the instruction methods used in the three schools.

Results

Frequency of Success on the Fraction Pie and Fraction Sort Tasks

All of the students correctly identified the circular quarter with equal pieces in the Fraction Sort task, see Figure 2b. However, only 69.3% identified the quarter in the Fraction Pie task, see Figure 2a. While 27.3% identified the 1/6 in the Fraction Pie task, no student correctly identified the 1/6 piece of the Fraction Pie who had not correctly identified the ¼ piece of the same diagram. All but two students could identify 1/6 in the equal parts diagram, see Figure 2c, and neither of those them could identify the 1/6 in the Fraction Pie task.

Types of Strategies Used by the Students on the Fraction Sort Task

The students' correct explanations of the two equal-area examples, see Figure 2b and 2c can be categorised into six types of responses: 1) a count of both shaded and non-shaded parts with a mention of equal sized pieces; 2) a double count without reference to the size of the pieces; 3) a reference to one quantity, for example, there are six there; 4) just a statement that it is a quarter or a sixth; 5) a half of a half; and 6) the usual quarter. The frequency of the responses is reported in Table 1. A double count, a count for all the pieces and a count for the shaded piece, for example, *there's four (six) there and one is coloured in so it's a quarter (sixth)* was used by many students on both equal-parts cards.

Table 1
Types of correct explanations for equal-area Fraction Sort cards (1/4: n=88, 1/6: n=86)

Fraction sort task	Size noted, two counts	double count	One count	Stated	Half of a half	usual quarter
Equal-parts quarter	1	64	5	13	4	1
Equal-parts sixth	2	70	1	13	0	0

The frequency of the students' answers to the non-equal-parts rectangular-sixth card, see Figure 2d, was: one sixth (44); one quarter (25); two thirds (1); and other (18). Using the categories from the research literature these would be interpreted as: correct (one sixth), count-and-match misconception (one quarter), odd (two thirds), and not count-and-match misconception (other). However, of the 25 students who placed the card in the quarter pile, three students added a qualification about size, *but not equal quarters*, and one student said it was both *a sixth and a quarter* but put it in the quarter pile. Conversely, there was one student who placed the card in the *other* pile because he had identified it as a fourth, and explained, *a fourth is not a quarter*. We classify that explanation as indicating a count-and-match misconception, despite the choice of *other*.

Types of Strategies Used on the Fraction Pie Task

The correct strategies used in identifying the quarter and the sixth in the Fraction Pie task can be categorised as: 1) looks like a quarter/sixth; 2) imagines line/s extending across right to left (or left to right) and ignores the line/s already there; 3) half of a half; 4) there's three on one side so that would be six altogether; 5) imagined the left hand side as the right hand side; 6) iterated a single piece; and 7) identified a right angle. The frequency of these responses is reported in the Table 2.

Table 2
Types of correct explanations for the Fraction Pie task (1/4: n=61, 1/6: n=24)

Fraction Pie	Looks like	¼ line extend	½ of ½	3 so 6	lhs as rhs	iterate	Right angle	other
quarter	7	30	18	0	0	0	4	2
sixth	0	0	0	9	11	3	0	1

The incorrect strategies used to identify the quarter or the sixth include: 1) 1/5 (or 1 out of 5) because there are 5 pieces; 2) 1/5 (or 1 out of 5) because there are five pieces, *but not an equal fifth*, or an answer of *both ¼ and a fifth*; 3) 2/5 because there are 5 pieces; 4) 2/7 or 1/7 by extending the sixth lines on the right hand side back across to the left hand side and not ignoring the quarters line; 5) 1/2 because there are two pieces on the left hand side or 1/3 because there are three pieces on the right hand side; 6) many different answers (1/2, 1/3, 1/5, 1/6, 1/7, 1/8, 2/3, .7, and don't know) because *it is nearly 1/4*; *half a quarter*; or *point 7 of a quarter*; 7) iterating the single piece imprecisely; and 8) other, including *don't know*. The frequency of responses is reported in Table 3.

Table 3
Types of incorrect explanations for Fraction Pie task (1/4: n=27, 1/6: n=64)

Fraction Pie	1/5: 5 pieces	1/5: 5 pieces, but…	2/5: 5 pieces	2/7, 1/7: lines	1/3, 1/2 : a side	Nearly a 1/4	iterates	other
quarter	13	3	1	3	1	0	0	6
sixth	11	2	3	2	8	24	3	11

Eight of the thirteen students who had demonstrated the count-and-match misconception, *1/5, because there are 5 pieces* for the quarter, repeated this for the sixth and two of the thirteen modified their second answer to 2/5. Of the 24 students who tried to compare part B to the quarter, four of them answered 1/5. These four students do not have the count-and-match misconception, despite giving the answer designated in the literature to identify it.

Discussion and Conclusion

The tasks reported on here are traditional part-whole tasks requiring the identification of shaded or labelled parts of a static diagram. While 30.7% of the students were unable to identify the quarter in the Fraction Pie task, all of them had identified the quarter in the equal parts diagram. It can be concluded that a lack of traditional part-whole knowledge of a quarter in a circular model was not a contributing factor to the students' performance on the Fraction Pie task. Similarly, all but two students successfully identified the sixth in the equal-parts diagram. Both of them miscounted the number of parts but neither of them could successfully identify the sixth in the Fraction Pie task. However, for the other 62 students, a lack of traditional part-whole knowledge of a sixth in a circular model was probably not a contributing factor to the students' performance on the Fraction Pie task.

The language used by the students to describe the equal-parts diagrams illustrates Kieren's pedagogical concern about part-whole contexts. The majority of students used a double count of the shaded pieces and all the pieces. This type of description, *a sixth, because there's six there and one's coloured in*, is not incorrect if there is a shared understanding between the student and the interviewer, that the assumption, *given equal parts…*, is implicit. However, the same verbal explanation can be used if a count-and-match misconception is present, and the assumption, given equal parts, absent. It is not possible in the equal-area parts of the Fraction Sort task to categorise students' responses as indicating the count-and-match misconception or not because the same explanation can be used to illustrate either understanding. This also works in reverse. If we cannot hear in such a response from a student whether a count-and-match misconception is present, then it is reasonable to assume that if a teacher were to use such an explanation then a child with a count-and-match misconception would not be cued into attending to the size of the pieces. Instead, the student would hear confirmation of what they were already thinking, and the misconception would go undetected.

Non-equal-parts fraction-area-models have previously been identified in the literature as part-whole tasks that enable the identification of the count-and-match misconception, termed part-whole discrete thinking by Saxe et al. (2005). The use of a probing interview in the current research has enabled not only another confirmation of the existence of the count-and-match misconception, but also a distinction to be drawn between a pure form of the misconception, *1/5, because there are five pieces*, and a modified version of the misconception with the qualification, *but not equal fifths*, or a different fifth (2/5) is named. Further to the results from the literature (Saxe et al., 2005; Stewart, 2005; Clarke et al., 2007) it was also evident that not all answers of 1/5 indicated a count-and-match misconception as four of the 24 students who had attempted to compare part B of the Fraction Pie to the known quarter also answered 1/5 but were not demonstrating the misconception.

The count-and-match strategy was evident, but its use was not uniform across all the tasks. The pure count-and-match strategy was present in each of the three tasks: rectangle non-equal parts 1/6 (29.5%), the quarter in the fraction pie (14.8%), and the sixth in the fraction pie (12.5%). While eight students used pure count-and match strategies in both the quarter and the sixth in the Fraction Pie task only one student also did that for the rectangular non-equal parts 1/6 in the Fraction Sort task.

Using Kieren's sub-constructs of measure, operator, quotient and ratio to analyse the students thinking on the traditional part-whole tasks can add to the part-whole analysis above. The measure sub-construct is more than just fraction number lines. In Kieren's later work (1993) this sub-construct encompasses many contexts using models with continuous quantities such as area. It is easier to introduce this context with improper fractions, which can represent units which have not measured something exactly and created a leftover that needs to be described. In the single-whole context the unshaded pieces in the traditional part-whole diagram represent what wasn't needed, and the shaded part is the leftover, or incomplete unit, that needs to be described. Many of the correct strategies for identifying the quarter in the Fraction Pie could be translated into this context. Recognising that a part looks like a quarter; is the half of a half; or that the whole can have four equal pieces if the line is extended across to the right hand side and the lines there ignored, are all strategies equally applicable to part-whole understanding and measure understanding. Giving a measure context, rather than presenting a static diagram can enable this other set of explanations to be used.

Another strategy used by the students in the Fraction Pie task can be analysed in both a part-whole and a quotient context. The students who extended the sixth lines back across the left hand side could be said to be trying to make equal parts. If they had ignored the quarter line distracter, they would have been successful; instead they made seven parts rather than six, and gave answers of 2/7 and 1/7. This approach can be seen as mathematically correct, but poorly executed. We note that they had to use visualisation as they could not draw on the task card. All three of the students who used this strategy correctly answered 1/6 for the rectangular non-equal-parts Fraction Sort card. Trying to make equal parts, as a strategy, would appear to be easier with the rectangle diagram of 1/6. The question for the rectangular 1/6 does not have to be, *what fraction is shaded?* (traditional part-whole), but rather, *given this start, how many people can share this cake fairly, and how much will they get?* The quotient sub-construct may be a way of turning their extending-the-lines-across *strategy* into a trying-to-make-equal-parts *explanation* of a quotient *context*. It is a context common in the Dutch curriculum (Streefland, 1991).

There was one strategy which did not fit easily with a part-whole context, but which can be explained using the operator sub-construct. Some students tried to compare part B with the quarter they had identified in part A of the Fraction Pie task. One difficulty for the students was that none of them could describe this relationship, 2/3 of a quarter, accurately. The closest description was by a student who explained that he had been trying to work out point seven of a quarter. The second difficulty was with the calculation of the stated fraction of a quarter. The range of answers described in the results section demonstrates that the students had difficulty calculating a re-sized part to match part B. This approach was not mathematically incorrect, and it is best understood within the operator context.

The use of traditional part-whole definitions has been described in this study. While not explored here, a change in phrasing from there's twelve there and two are coloured in to there's one coloured in for every six may have improved the students' attempts at equivalent representations on other cards in the Fraction Sort task. This language would be privileged in a ratio context.

Part-whole diagrams, language and misconceptions have not disappeared from classrooms, just as fraction algorithms still exist. But for Kieren (1993) they exist as descriptions of practice. His four-part model – measure, quotient, operator and ratio, is an ideal model of contexts and understandings that could enable a curriculum and explanations that support the varied ways in which fractions can be understood. This study has demonstrated that it is possible to analyse traditional part-whole tasks using Kieren's four-part model. On some occasions this can be in addition to traditional part-whole explanations, which exist outside Kieren's model. However, in the operator example explored above, Kieren's four-part model provided a context to analyse a mathematically correct, but poorly executed approach that was not well explained by traditional part-whole definitions.

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